

Crypto-Hermiticity of nonanticommutative theories

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Abstract

We note that, though nonanticommutative deformations of Minkowski supersymmetric theories do not respect the reality condition and seem to lead to non-Hermitian Hamiltonians H , the latter belong to the class of crypto-Hermitian (or quasi-Hermitian) Hamiltonians having attracted recently a considerable attention. They can be made manifestly Hermitian via the similarity transformation $H \rightarrow e^R H e^{-R}$ with a properly chosen R . The deformed model enjoys *the same* supersymmetry algebra as the undeformed one though it is difficult in some cases to write explicit expressions for a half of supercharges. The deformed SQM models make perfect sense. It is not clear whether it is also the case for NAC Minkowski field theories — the conventionally defined S -matrix is not unitary there.

1 Introduction

Supersymmetric models with nonanticommutative (NAC) deformations [1] have recently attracted a considerable interest. The main idea is that the odd superspace coordinates θ^α and $\bar{\theta}^{\dot{\alpha}}$ are not treated as strictly anticommuting anymore, but involve non-vanishing anticommutators [2] ¹. In original Seiberg's paper and in many subsequent works (see e.g. [3, 4] and references therein), the deformation is performed in Euclidean rather than Minkowski space-time. The reason is that in Minkowski space it seems impossible to preserve both supersymmetry and reality of the action after deformation, still retaining simple properties of the corresponding \star -product (e.g., associativity and nilpotency). As discussed in Ref. [1], Euclidean NAC theories are of interest in stringy perspectives. An interesting question is whether NAC theories are meaningful by themselves, leaving aside the issue of their relationships with string theory. In other words — whether it is possible to consistently define them in Minkowski space.

We argue that the answer to this question is at least partially positive. Namely, we will show that, for NAC theories put in finite spatial box, one can introduce a Hamiltonian with real spectrum and find a unitary finite time evolution operator. However, S -matrix obtained after projecting this operator onto conventionally defined $|in\rangle$ and $|out\rangle$ asymptotic states *is* not unitary.

Our consideration is based on the analysis of two SQM models — (i) an interesting 1-dimensional NAC model constructed in a recent paper of Aldrovandi and Schaposnik [5] and (ii) the model obtained from NAC Wess-Zumino model by dimensional reduction.

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¹In other words, the original Grassmann algebra of the odd coordinates is deformed into a *Clifford* algebra.

In Ref. [5], NAC deformations of the conventional Witten’s supersymmetric quantum mechanics (SQM) model [6] were studied in the chiral basis. In this case, the deformation operator commutes with the supercharge Q , but does not commute with \bar{Q} . However, Aldrovandi and Schaposnik noticed the presence of the second supercharge \bar{Q} that commutes with the Hamiltonian. On the other hand, Q and \bar{Q} seem not to be adjoint to each other and the deformed Hamiltonian seems to lack Hermiticity.

Our key observation [7] is that, in spite of having a complex appearance, this Hamiltonian is actually Hermitian in disguise. One can call it “crypto-Hermitian” (or “cryptoreal”). It belongs to the class of Hamiltonians having attracted recently a considerable attention (the Hamiltonians of this kind are known since mid-seventies [8], but these studies received great impetus after the beautiful paper [9]). One of the simplest examples is

$$H = \frac{p^2 + x^2}{2} + igx^3. \quad (1)$$

In spite of the manifestly complex potential, it is possible to endow the Hamiltonian (1) with a properly defined Hilbert space such that the spectrum of H is real. The clearest way to see this is to observe the existence of the operator R such that the conjugated Hamiltonian

$$\tilde{H} = e^R H e^{-R} \quad (2)$$

is manifestly self-adjoint [10]. The explicit form of R for the Hamiltonian (1) is ²

$$R = g \left(\frac{2}{3} p^3 + x^2 p \right) - g^3 \left(\frac{64}{15} p^5 + \frac{20}{3} p^3 x^2 + 4 p x^4 - 6 p \right) + O(g^5). \quad (3)$$

The rotated Hamiltonian is

$$\tilde{H} = \frac{p^2 + x^2}{2} + g^2 \left(3 p^2 x^2 + \frac{3 x^4}{2} - \frac{1}{2} \right) + O(g^4). \quad (4)$$

The (real) spectrum of \tilde{H} (and H) can be found to any order in g in the perturbation theory, and also non-perturbatively.

We will see that in the case of the Aldrovandi-Schaposnik Hamiltonian, there also exists the operator R making the Hamiltonian Hermitian. The rotated supercharges $e^R Q e^{-R}$ and $e^R \bar{Q} e^{-R}$ are Hermitian-conjugated. Such an operator must exist also for the NAC WZ model.

2 Aldrovandi-Schaposnik model

The simplest SQM model [6] involves a real supervariable

$$X(\theta, \bar{\theta}, t) = x(t) + \theta \psi(t) + \bar{\psi}(t) \bar{\theta} + \theta \bar{\theta} F(t). \quad (5)$$

²Actually, what is written here is the Weyl symbol of the operator R . The expression for a contribution to the quantum operator corresponding to a monomial $\sim p^n x^n$ in its Weyl symbol is a properly symmetrized structure, $px \rightarrow (1/2)(\hat{p}x + x\hat{p})$, $x^2 p \rightarrow (1/3)(x^2 \hat{p} + \hat{p}x^2 + x\hat{p}x)$, etc.

The action is

$$S = - \int dt d^2\theta \left[\frac{1}{2} (DX)(\bar{D}X) + V(X) \right], \quad (6)$$

with the convention $\int d^2\theta \theta \bar{\theta} = 1$. Here $V(X)$ is the superpotential and D, \bar{D} are covariant derivatives. Bearing in mind the deformation coming soon, we will choose their left chiral basis representation

$$D = \frac{\partial}{\partial \theta} - 2i\bar{\theta} \frac{\partial}{\partial t}, \quad \bar{D} = -\frac{\partial}{\partial \bar{\theta}}. \quad (7)$$

Here $t = \tau - i\theta\bar{\theta}$ and τ is the real time coordinate of the central basis. Asymmetry between D and \bar{D} makes the Lagrangian following from (6) complex,

$$L = -i\dot{x}F - \frac{\partial V(x)}{\partial x}F + \frac{1}{2}F^2 + i\bar{\psi}\dot{\psi} + \frac{\partial^2 V(x)}{\partial x^2}\bar{\psi}\psi, \quad (8)$$

but one can easily make it real, rewriting it in terms of $\tilde{F} = F - i\dot{x}$ and subtracting a total derivative. This corresponds to going over to the central basis from the chiral one.

The deformation is introduced by postulating non-vanishing anticommutators

$$\{\theta, \theta\} = C, \quad \{\bar{\theta}, \bar{\theta}\} = \bar{C}, \quad \{\theta, \bar{\theta}\} = \tilde{C}. \quad (9)$$

The deformed action involves star products,

$$S = - \int dt d^2\theta \left[\frac{1}{2} (D \star X) \star (\bar{D} \star X) + V_\star(X) \right], \quad (10)$$

where

$$X \star Y = \exp \left\{ -\frac{C}{2} \frac{\partial^2}{\partial \theta_1 \partial \theta_2} - \frac{\bar{C}}{2} \frac{\partial^2}{\partial \bar{\theta}_1 \partial \bar{\theta}_2} - \frac{\tilde{C}}{2} \left(\frac{\partial^2}{\partial \theta_1 \partial \bar{\theta}_2} + \frac{\partial^2}{\partial \bar{\theta}_1 \partial \theta_2} \right) \right\} X(1)Y(2) \Big|_{1=2} \quad (11)$$

and $V_\star(X)$ is obtained from $V(X) = \sum_n c_n X^n$ by substituting $X^2 \rightarrow X_\star^2 \equiv X \star X$, $X^3 \rightarrow X_\star^3 \equiv X \star X \star X$, etc in its Taylor expansion. The star product is associative.

The component expression for the deformed Lagrangian is the same as in Eq. (8), with $V(x)$ being substituted by [5, 12]

$$\tilde{V}(x, F) = \int_{-1/2}^{1/2} d\xi V(x + \xi c F), \quad (12)$$

where

$$c^2 = \tilde{C}^2 - C\bar{C} \quad (13)$$

is the relevant deformation parameter. If \bar{C} is conjugate to C and \tilde{C} is real, c^2 is also real. Note, however, that one may, generally speaking, lift the condition that θ and $\bar{\theta}$ are conjugate to each other, in which case C, \bar{C} and \tilde{C} can take arbitrary values. We still require the reality of c^2 . The crypto-Hermiticity of the deformed Hamiltonian discussed below is fulfilled under this condition.

In the simplest nontrivial case, $V(X) = \lambda X^3/3$,

$$\tilde{V}(x, F) = \frac{\lambda x^3}{3} + \frac{\lambda c^2 x F^2}{12} . \quad (14)$$

The corresponding canonical Hamiltonian is

$$H = \frac{p^2}{2} + i \frac{\partial \tilde{V}}{\partial x} p - \frac{\partial^2 \tilde{V}}{\partial x^2} \bar{\psi} \psi , \quad (15)$$

with $p = -iF$. The deformed Lagrangian and Hamiltonian look inherently complex. Obviously, the complexities now cannot be removed by simply going from the chiral to the central basis.

In the chiral basis, the supercharges are represented by the following superspace differential operators,

$$Q = \frac{\partial}{\partial \theta} , \quad \bar{Q} = -\frac{\partial}{\partial \bar{\theta}} - 2i\theta \frac{\partial}{\partial t} . \quad (16)$$

Note that the star product operator (11) still commutes with Q (in other words, the Leibnitz rule $Q \star (X \star Y) = (Q \star X) \star Y + X \star (Q \star Y)$ still holds), but not with \bar{Q} . That means that the deformed action (10) is still invariant with respect to the supersymmetry transformations generated by Q , but not \bar{Q} . The Q -invariance implies the existence of the conserved Nöther supercharge whose component phase space expression is simply

$$Q = \psi p . \quad (17)$$

As was observed in [5], there is another Grassmann-odd operator commuting with the Hamiltonian. It reads

$$\bar{Q} = \bar{\psi} \left(p + 2i \frac{\partial \tilde{V}}{\partial x} \right) . \quad (18)$$

The standard SUSY algebra

$$Q^2 = \bar{Q}^2 = 0, \quad \{Q, \bar{Q}\} = 2H \quad (19)$$

holds, but, naively, \bar{Q} is not adjoint to Q and H is not Hermitian.

Let us show now that the Hamiltonian (15) is in fact cryptoreal. Consider for simplicity only the case (14). We have,

$$H = \frac{p^2}{2} + i\lambda p x^2 - i\beta p^3 - 2\lambda x \bar{\psi} \psi , \quad (20)$$

where $\beta = \lambda c^2/12$.

It is convenient to treat λ and β on equal footing and to get rid of the complexities $\sim ipx^2$ and $\sim ip^3$ simultaneously. The operator R doing this job is

$$R = -\frac{\lambda x^3}{3} + \beta x p^2 - 2\lambda \beta x^2 \bar{\psi} \psi + \dots , \quad (21)$$

where the dots stand for the terms of the third and higher order in λ and/or β . The conjugated Hamiltonian is

$$\tilde{H} = e^R H e^{-R} = \frac{p^2}{2} - 2\lambda x \bar{\psi} \psi + \frac{1}{2}[\lambda^2 x^4 + 3\beta^2 p^4] + \frac{1}{2}\lambda\beta + O(\lambda^3, \beta^3, \lambda^2\beta, \lambda\beta^2). \quad (22)$$

It is Hermitian. The rotated supercharges are

$$\begin{aligned} \tilde{Q} &= e^R Q e^{-R} = \psi[p - i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p - \beta^2 p^3 + \dots], \\ \tilde{\bar{Q}} &= e^R \bar{Q} e^{-R} = \bar{\psi}[p + i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p + 3\beta^2 p^3 + \dots]. \end{aligned} \quad (23)$$

We observe that they are still not adjoint to each other. To make them mutually adjoint to the considered order in β, λ , one should add to the operator R one more term

$$R \Rightarrow \hat{R} = R - 2\beta^2 p^2 \bar{\psi} \psi. \quad (24)$$

It is easy to see that this modification does not change the rotated Hamiltonian in the considered order, but ensures the rotated supercharges to be manifestly adjoint to each other

$$\begin{aligned} \hat{Q} &= e^{\hat{R}} Q e^{-\hat{R}} = \psi[p - i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p + \beta^2 p^3 + \dots], \\ \hat{\bar{Q}} &= e^{\hat{R}} \bar{Q} e^{-\hat{R}} = \bar{\psi}[p + i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p + \beta^2 p^3 + \dots]. \end{aligned} \quad (25)$$

By construction, the operators $\hat{Q}, \hat{\bar{Q}}$ and \tilde{H} satisfy the standard algebra (19). We see that the requirement of the mutual adjointness of supercharges is to some extent more fundamental than that of the Hermiticity of the Hamiltonian — the latter does not strictly fix the rotation operator R while the former does.

One can be convinced, order by order in β, λ , that complexities in H can be successfully rotated away also in higher orders (with simultaneously restoring the mutual conjugacy of the supercharges), and this is also true for higher powers $N > 3$ in $V(X) \sim X^N$ and hence for any analytic superpotential ³.

3 NAC Wess-Zumino model

The first example of an anticommutative deformation of a supersymmetric field theory was considered in Ref. [1]. Seiberg took the standard Wess-Zumino model

$$\begin{aligned} \mathcal{L} &= \int d^4\theta \bar{\Phi} \Phi + \left[\int d^2\theta \left(\frac{m\Phi^2}{2} + \frac{g\Phi^3}{3} \right) + \text{c.c.} \right] \\ &\equiv |\partial_\mu \phi|^2 + i\bar{\psi} \hat{\partial} \psi - |F(\phi)|^2 + [F'(\phi)\psi^2 + H.c.] \end{aligned} \quad (26)$$

with $F(\phi) = m\phi + g\phi^2$ and deformed it by introducing the nontrivial anticommutator

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}, \quad (27)$$

$C^{\alpha\beta} = C^{\beta\alpha}$, in the assumption that all other (anti)commutators vanish,

$$\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = [\theta^\alpha, x_\mu^L] = [\bar{\theta}^{\dot{\alpha}}, x_\mu^L] = [x_\mu^L, x_\nu^L] = 0. \quad (28)$$

³It would be worth being aware of the full analytic proof of this.

Note that this all was written in the *chiral* basis, $x_\mu^L = x_\mu^{\text{central}} + i\theta\sigma_\mu\bar{\theta}$. In Ref. [1], the space x_μ was assumed to be Euclidean. We will work in Minkowski space, however, and will not be scared by the appearance of complexities at intermediate steps. The Minkowski space deformation (27), (28) is analogous to the SQM deformation (9) with $\bar{C} = \tilde{C} = 0$.

The anticommutator (27) introduces a constant self-dual tensor, which explicitly breaks Lorentz invariance. However, the deformed Lagrangian expressed in terms of the component fields proves still to be Lorentz invariant. Indeed, it is easy to find that the kinetic term $\int d^4\theta \bar{\Phi}\Phi$ is undeformed and the only extra piece comes from

$$\Delta\mathcal{L} = \frac{g}{3} \int d^2\theta \Phi * \Phi * \Phi - \frac{g}{3} \int d^2\theta \Phi^3 = -\frac{g}{3} \det \|C\| F^3. \quad (29)$$

It depends only on the scalar $\det \|C\|$ and is obviously Lorentz invariant. Adding the usual terms $F(m\phi + g\phi^2) + \bar{F}(m\bar{\phi} + \bar{g}\bar{\phi}^2)$ coming from superpotential and $F\bar{F}$ from the kinetic term, and expressing F and \bar{F} via ϕ and $\bar{\phi}$, we see that the undeformed potential $|m\phi + g\phi^2|^2$ acquires an extra holomorphic contribution $\propto g(m\bar{\phi} + \bar{g}\bar{\phi}^2)^3$.

When $g = 0$, the undeformed model is free and so is deformed one. The interacting model is deformed, however, in a nontrivial way [13]. Contrary to our original hope [7], the spectrum is shifted. To see this explicitly, let us consider the dimensionally reduced system and assume that the fields do not depend on spatial coordinates. The reduced Hamiltonian is

$$H = \bar{\pi}\pi + \bar{\phi}\phi + g\phi^2\bar{\phi} + \bar{g}\bar{\phi}^2\phi + g\bar{g}\bar{\phi}^2\phi^2 - (1 + 2g\phi)\psi_1\psi_2 - (1 + 2\bar{g}\bar{\phi})\bar{\psi}_2\bar{\psi}_1 + \beta(\bar{\phi} + \bar{g}^2\bar{\phi}^2)^3 \quad (30)$$

with $\bar{\psi}_\alpha \equiv \partial/\partial\psi_\alpha$ and $\beta = g \det \|C\|/3$ being the deformation parameter. For simplicity, we have set $m = 1$.

The wave functions for this Hamiltonian have four components, being represented as

$$\Psi(\phi, \bar{\phi}, \psi_\alpha) = A(\bar{\phi}, \phi) + B_\alpha(\bar{\phi}, \phi)\psi_\alpha + C(\bar{\phi}, \phi)\psi_1\psi_2. \quad (31)$$

In the undeformed case, the Hamiltonian (30) admits conserved supercharges

$$Q_\alpha = \pi\psi_\alpha + i\epsilon_{\alpha\gamma}\bar{\psi}_\gamma(\bar{\phi} + \bar{g}\bar{\phi}^2), \\ \bar{Q}_\beta = \bar{\pi}\bar{\psi}_\beta - i\epsilon_{\beta\delta}\psi_\delta(\phi + g\phi^2) \quad (32)$$

with $\epsilon_{12} = 1$. They satisfy the usual $\mathcal{N} = 2$ SQM algebra

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0, \quad \{Q_\alpha, \bar{Q}_\beta\} = H\delta_{\alpha\beta} \quad (33)$$

The spectrum of the undeformed Hamiltonian involves a single vacuum state, while the excited states come in quartets: there is a quartet of states of energy 1, two quartets of energy 2, etc. One can check by an explicit perturbative calculation (It is a fourth order calculation. The relevant graphs are shown in Fig. 1) that the ground state of the perturbed Hamiltonian still has the zero energy, while the energies of the excited states are shifted. For example, for the first excited quartet,

$$\Delta E_1 = -\frac{155}{36}\beta g^3 + \text{higher order terms}. \quad (34)$$

If βg^3 is real, the energy shift is also real.

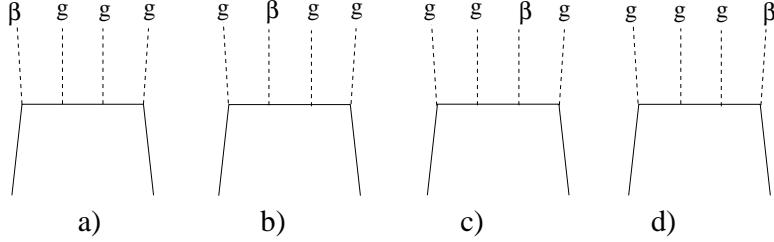


Figure 1: Graphs contributing to the energy shift $\propto \beta g^3$

It is important that the spectrum keeps its structure dictated by the supersymmetry (32) and consists of degenerate quartets. This means that the deformed model still enjoys full supersymmetry of the undeformed theory and the algebra (32) still holds.

Speaking of the supercharge Q_α , it is still given by the expression in Eq.(32), which commutes with the deformed Hamiltonian. On the other hand, the commutator of the undeformed supercharge \bar{Q}_α with the deformed Hamiltonian does not vanish. In contrast to the Aldrovandy-Schaposnik model considered in the previous section, we cannot write a simple expression for the deformed supercharge. By no means can it be obtained by complex conjugation of the supercharge Q_α . Indeed, a pair of complex conjugate supercharges would mean Hermiticity of Hamiltonian, but the Hamiltonian (30) is not manifestly Hermitian. The fact that its spectrum is real (when β, g are real) tells, however, that the Hamiltonian is crypto-Hermitian in the same sense as the Aldrovandy-Schaposnik Hamiltonian is. In particular, the operator R rotating the Hamiltonian to the manifestly Hermitian form should exist.

Even though explicit expressions for \bar{Q}_α are not known, one can argue that the quartet supersymmetric structure of the spectrum must hold without making explicit calculations. It can be reconstructed (at least, perturbatively ⁴) using only Q_α and not \bar{Q}_α . Indeed, for each supersymmetric quartet of the eigenstates of the free Hamiltonian H_0 , a member Ψ annihilated by the action of \bar{Q}_α , but not Q_α , can be chosen. Three other members of the quartet are $Q_{1,2}\Psi$ and $Q_1Q_2\Psi$. Let $\tilde{\Psi}$ be the corresponding eigenstate of the full Hamiltonian (when β and g are small, one can be sure that such state exists). Then $\tilde{\Psi}$, $Q_\alpha\tilde{\Psi}$, and $Q^2\tilde{\Psi}$ represent a quartet of degenerate eigenstates of the interacting deformed Hamiltonian. Once the states are known, the matrix elements of \bar{Q}_α can be defined to be equal to the corresponding matrix elements in the free undeformed basis multiplied by $\sqrt{E_n^{\text{exact}}/E_n^{\text{free}}}$.

What conclusions concerning NAC field theories can be made on the basis of this analysis? If we put the theory in a finite spatial box and be interested in the spectrum of the Hamiltonian thus obtained, its properties should be similar to the properties of the dimensionally reduced Hamiltonian:

- The ground state energy(ies) is(are) still zero (if supersymmetry is not spontaneously broken) and the 2^N degeneracy of the excited spectrum states should be kept.
- For certain values of the deformation parameters and the couplings, the spectrum

⁴It would be very interesting to study the spectrum of the deformed Hamiltonian numerically. One cannot exclude a possibility that exceptional points [14] in the space of couplings appear such that the supersymmetric structure of the spectrum would be lost for large enough values of β, g .

of the deformed Hamiltonian should enjoy crypto-Hermiticity property.

However, the main question we usually ask in field theories is not what are the spectra of their finite box Hamiltonians, but what are their S -matrices — the matrix elements of the evolution operator between the asymptotic $|in\rangle$ and $|out\rangle$ states. For NAC theories, the complexity of Lagrangian strikes back at this point: for *conventionally* defined asymptotic states, the S -matrix for, say NAC Wess-Zumino model is not unitary (see Ref. [13] for more detailed discussion).

This means that NAC theories obtained by deformation of *interacting* SUSY theories cannot be attributed a *conventional* physical meaning. More studies of this question are necessary. Maybe even if S -matrix of the theory is not unitary, unitarity of its finite time finite box evolution operator (that follows from crypto-Hermiticity of the Hamiltonian) suffices to make the theory meaningful? A positive answer to this question would mean a breakthrough in understanding not only NAC theories, but also theories with higher derivatives in the Lagrangian. In Ref. [15], we argued that the fundamental Theory of Everything may be a theory of this kind. We address the reader to this paper and also to the papers [16] for discussions and speculations on this subject.

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